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Criticality of the Potts ferromagnet in Migdal–Kadanoff-like hierarchical lattices

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Abstract. Within the real space renormalisation group framework, we discuss the critical point and exponent ν of the Potts ferromagnet in b-sized Migdal-Kadanoff-like hierarchical lattices. Both $b \to \infty$ and $b \to 1$ limits are exhibited. The important discrepancies that might exist between the exact results for d-dimensional hierarchical lattices and d-dimensional Bravais lattices are illustrated.

1. Introduction

The study of the criticality of magnetic models (e.g., the Potts model) on d-dimensional Bravais lattices is frequently replaced, within some real space renormalisation group (RG) techniques, by the study of d_f-dimensional hierarchical lattices which satisfy $d_f \to d$ in the limit of large cells ($b \to \infty$). (Here $d_f \equiv$ intrinsic fractal dimensionality $\equiv \ln N_b / \ln b$, where N_b is the number of bonds of the two-rooted graph whose iteration generates the hierarchical lattice, and b is the chemical distance between its roots [1, 2]. It is important to evaluate, both qualitatively and quantitatively, the benefits as well as the restrictions of such procedures (see [1-3] and references therein). A very simple and commonly used framework is the Migdal-Kadanoff (or bond-moving) framework [4] which is based on the so-called diamond hierarchical lattice. Here we generalise this procedure through a comprehensive discussion of the criticality of the q-state Potts ferromagnet. This constitutes a clear illustration of analogies and discrepancies between Bravais and hierarchical lattices. Some of the results appearing in [1] and [5] ($q = 2$ and the $b \to 1$ limit for $q = 1$, respectively) are herein recovered as particular cases.

2. Model and formalism

We consider the b-sized, d-dimensional diamond (tress) hierarchical lattice; it is defined through infinite iteration of a two-rooted graph which consists in an array of $b^{d-1}(b)$ strings in parallel (series), each of them constituted by $b(b^{d-1})$ bonds in series (parallel). Typical lattices of this type are presented in figure 1. Two important topological properties are verified: (i) for all b and d and both diamond and tress types, the intrinsic fractal dimensionality is given by $d_f = \ln b^d / \ln b = d$ and (ii) for arbitrary fixed b and $d = 2$, and only in this case, the diamond and tress hierarchical lattices are dual of each other.

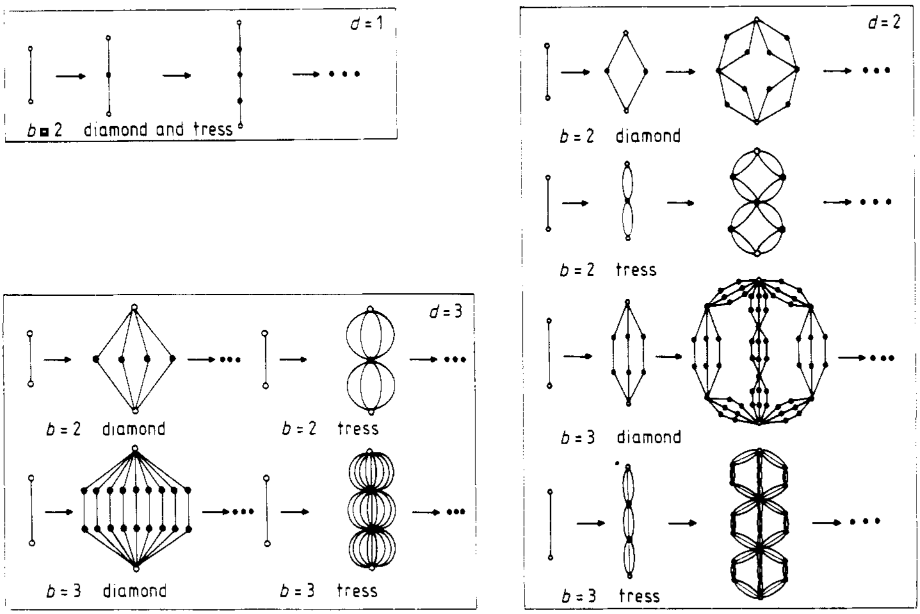

Figure 1. Typical b-sized d-dimensional two-rooted graphs and their corresponding hierarchical lattices (○ and ● denote the roots and internal sites respectively).

Each bond of these lattices represents the elementary Potts interaction, whose Hamiltonian is given by $\mathscr{H} = -qJ\delta_{\sigma_i,\sigma_j}$ ($J > 0$; the site variables σ_i and σ_j take the values $1, 2, \ldots, q$). We introduce the convenient variable

$$t \equiv [1 - \exp(-qJ/k_B T)]/[1 + (q-1)\exp(-qJ/k_B T)]$$

(named thermal transmissivity [6]). Both diamond and tress graphs are reducible in series and parallel operations. Therefore the corresponding transmissivities (denoted G_D and G_T respectively) can be easily calculated [6], thus yielding

$$G_D(t; b, d, q) = \left[1 - \left(\frac{1-t^b}{1+(q-1)t^b}\right)^{b^{d-1}}\right]\left[1 + (q-1)\left(\frac{1-t^b}{1+(q-1)t^b}\right)^{b^{d-1}}\right]^{-1} \tag{1}$$

and

$$G_T(t; b, d, q) = \left\{\left[1 - \left(\frac{1-t}{1+(q-1)t}\right)^{b^{d-1}}\right]\left[1 + (q-1)\left(\frac{1-t}{1+(q-1)t}\right)^{b^{d-1}}\right]^{-1}\right\}^b. \tag{2}$$

Let us now focus on the diamond case (the tress case is strictly analogous). We renormalise, for fixed d and q, a b-sized graph into a b'-sized one. Within this approach (hereafter referred to as $\mathrm{RG}_{bb'}$) the recursive relation is given by

$$G_D(t'; b', d, q) = G_D(t; b, d, q). \tag{3}$$

This equation admits, for all (b, b', d, q) two trivial (stable) fixed points, namely $t = 0$ (paramagnetic phase; P) and $t = 1$ (ferromagnetic phase; F), as well as a critical (unstable) fixed point denoted $t^*_{bb'}$ which satisfies

$$G_D(t^*_{bb'}; b', d, q) = G_D(t^*_{bb'}; b, d, q). \tag{4}$$

The corresponding thermal critical exponent $\nu_{bb'}$ is given by

$$\nu_{bb'} = \frac{\ln (b/b')}{\ln (\lambda_b/\lambda_{b'})} \tag{5}$$

with

$$\lambda_b \equiv \left(\frac{\mathrm{d}G_\mathrm{D}(t;\, b, d, q)}{\mathrm{d}t}\right)_{t=t^*_{bb'}} \qquad \lambda_{b'} = \left(\frac{\mathrm{d}G_\mathrm{D}(t;\, b', d, q)}{\mathrm{d}t}\right)_{t=t^*_{bb'}}$$

3. Results

The critical point $t^*_{bb'}$ depends on (b, b', d, q). These dependences are illustrated in figure 2 (b-evolution of t^*_{b1} and $t^*_{b,b-1}$ for $d = q = 2$) and figure 3 (t^*_{21} as a function of (d, q)). The values obtained for t^*_{b1} are exact for the corresponding hierarchical lattices.

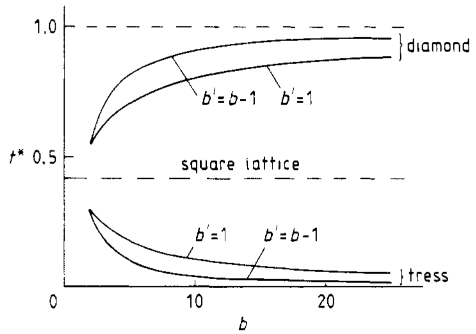

Figure 2. Critical points $t^*_{bb'}$ plotted against size b within the $\mathrm{RG}_{bb'}$ approach for $q = d = 2$, with $b' = 1$ and $b' = b - 1$, for both diamond and tress types; — · — denotes the exact result for the Ising ferromagnet in the square lattice.

The critical exponent $\nu_{bb'}$ depends on (b, b', d, q), but its value is one and the same for the diamond and tress cases. These dependences are illustrated in figure 4 (b-evolution of ν_{b1} and $\nu_{b,b-1}$ for $d = q = 2$) and figure 5 (ν_{21} as a function of (d, q)). The values obtained for ν_{b1} are exact for the corresponding hierarchical lattices. The d-dependence of ν_{21} at a fixed value of q deserves the following comments.

(i) For q high enough (q above $q_{\max} = 2$), ν_{21} presents, as a function of d, a minimum at a value of d (hereafter referred to as $d_{\min}$), and then increases again and reaches the value 1 in the $d \to \infty$ limit; $d^*_{\min}$ monotonically increases with increasing q and finally diverges in the $q \to \infty$ limit. The whole convergence is a non-uniform one. We verify that

$$\lim_{q\to\infty} \nu_{2,1} = 1/d \qquad d \geqslant 1 \tag{6}$$

which confirms the conjecture [2] that $\lim_{q\to\infty} \nu_{b,1} = 1/d_\mathrm{f}$. Also, $\lim_{d\to\infty} \lim_{q\to\infty} \nu_{21} = 0$ while $\lim_{q\to\infty} \lim_{d\to\infty} \nu_{21} = 1$.

(ii) For q low enough (q below $q_{\min} \simeq 0.215$), ν_{21} presents, as a function of d, a local maximum at a value of d (hereafter referred to as $d_{\max}$), and diverges in the $d \to 1$ limit; $d_{\max}$ monotonically increases from slightly below 2 to 2 while q decreases

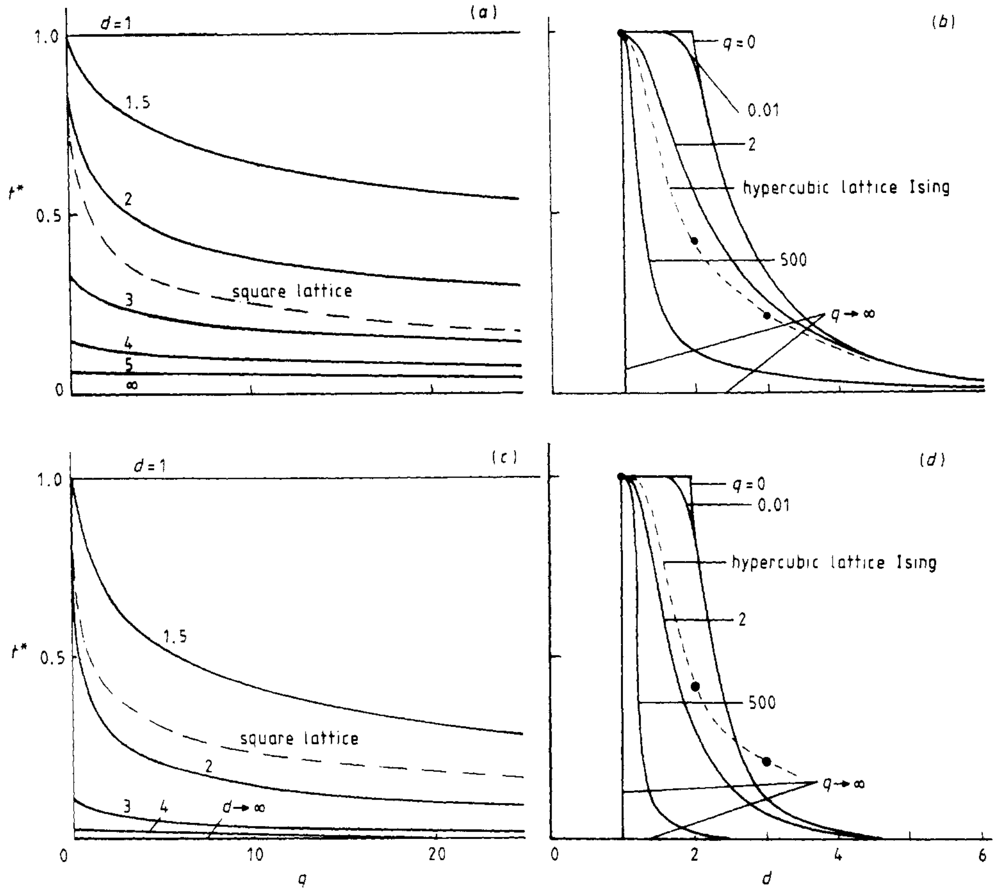

Figure 3. Dependence of the critical point t^* on q and d within the RG_{12} approach for (*a*), (*b*) diamond and (*c*), (*d*) tress. In (*a*) and (*c*) the exact results for square lattice has been included for comparison. In (*b*) and (*d*) the results corresponding to the Ising ferromagnet in the hypercubic lattice have been included for comparison, the broken line being a guide to the eye.

from q_{min} to 0; ν_{21} (d_{max}) monotonically increases from about 2.96 to infinity while q decreases from q_{min} to 0. Consistent with these observations, the $q = 0$ curve ν_{21} against d presents two branches: (a) in the interval $1 \leq d \leq 2$, ν_{21} presents a minimum at $d \simeq 1.5$ and $\nu_{21} \simeq 3.36$, and diverges in both the $d \to 1+0$ and $d \to 2-0$ limits and (b) for $d \geq 2$, ν_{21} monotonically decreases from infinity to one while d increases from 2 to infinity.

Points (i) and (ii) above have been verified for ν_{21}. Although we have not systematically checked, similar facts are expected for $\nu_{bb'}$. Summarising, three regimes are observed, as follows.

(i) $0 \leq q < q_{min}$: except for a local maximum in the neighbourhood of $d = 2$, the general trend of $\nu_{bb'}$ is to decrease from infinity to one while d increases from one to infinity.

(ii) $q_{min} \leq q \leq q_{max}$: $\nu_{bb'}$ monotonically decreases from infinity to one while d increases from one to infinity;

(iii) $q > q_{max}$: $\nu_{bb'}$ presents a minimum while d increases from one to infinity ($\nu_{bb'}$ diverges in the $d \to 1$ limit, and goes to one in the $d \to \infty$ limit).

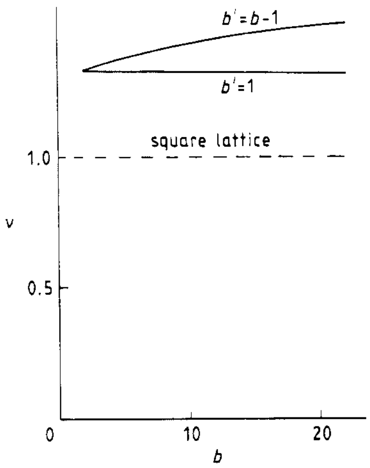

Figure 4. Critical exponent ν plotted against size b within the $\mathrm{RG}_{bb'}$ approach for $q = d = 2$, with $b' = 1$ and $b' = b - 1$, for both diamond and tress types (one and the same); — · — denotes the exact result for the Ising ferromagnet in the square lattice.

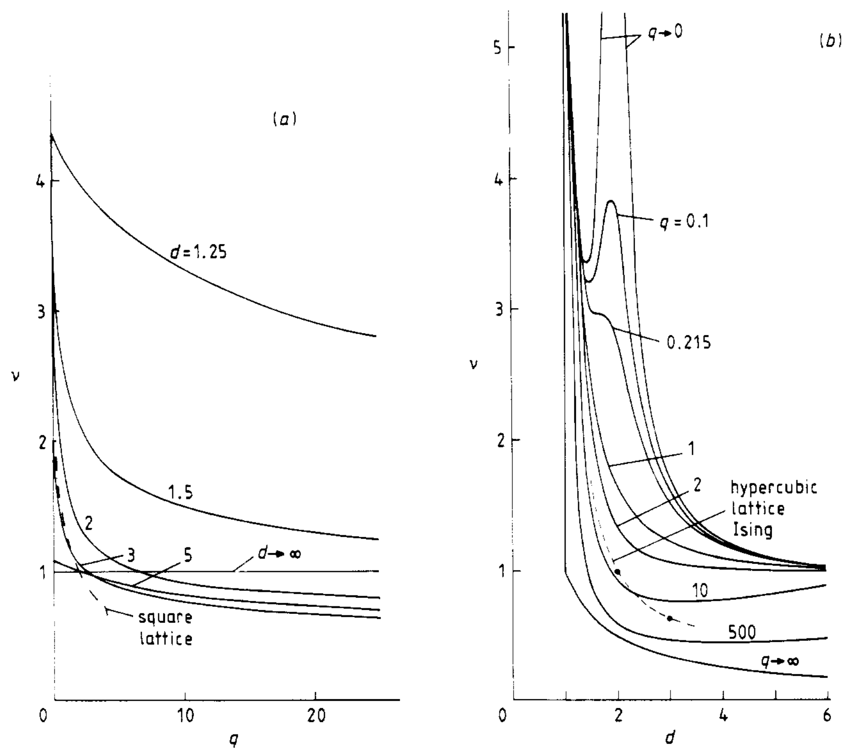

Figure 5. Dependence of the critical exponent ν on q and d within the RG_{12} approach (one and the same for both diamond and tress types). (a) ν against q for typical values of d (the exact result for square lattice has been included for comparison). (b) ν against d for typical values of q (the results corresponding to the Ising ferromagnet in the hypercubic lattice have been included for comparison, the broken line being a guide to the eye).

Another point which deserves comment is the $b \to \infty$ behaviour of $t^*_{bb'}$ and $\nu_{bb'}$. Our numerical results are consistent with the following behaviours.

(i) Diamond lattices: $t^*_{b,1} \sim 1-(d-1)\ln b/b\ (\forall q)$, and $t^*_{b,b-1} \sim 1-A(d,q)/b$, $A(d,q)$ being a pure number which satisfies $A(1,q)=0$ (similar laws are obtained for the tress lattices).

(ii) Diamond and tress lattices: $\nu_{b,1} \sim B(d)\ln b/\ln\ln b\ (\forall q)$, $B(d)$ being a pure number which decreases for increasing d; $\nu_{b,b-1}$ becomes almost independent of b and practically coincides, in the $b\to\infty$ limit, with $\nu_{21}(d,q)$ (see figure 5).

The result obtained for $\nu_{b,1}$ is in variance with the behaviour expected for lattices with finite critical temperature (i.e. $0<\lim_{b\to\infty,b'<b} t^*_{bb'}<1$). In such cases, finite-size scaling arguments [7] usually suggest, in the $b\to\infty$ limit, a logarithmic approach to a finite value.

Let us now turn our attention to a different type of limit, namely the differential one (i.e. $b'=1$ and $b=1+\mu$ with $\mu\to 0+$). We first notice that if we consider the hierarchical lattices generated by the b-sized d-dimensional generalised Wheatstone-bridge graphs (see [2] and references therein) with transmissivity denoted by G_W, we have, for all $(t;b,d,q)$,

$$G_D(t;b,d,q) \leqslant G_W(t;b,d,q) \leqslant G_T(t;b,d,q). \tag{7}$$

This is a trivial consequence of the fact that the transmissivity of *any* graph is a monotonously increasing function of the elementary transmissivity of any of its bonds, together with the fact that the breaking (collapsing) of all the 'transverse' bonds of the Wheatstone-bridge graph precisely yields the diamond (tress) graph [6]. It is then straightforward to verify that, in the $b\to 1$ limit, the RG_{b1} recursive relation is one and the same for both diamond and tress cases (and consequently for the Wheatstone-bridge case as well, as it is between the other cases), namely

$$t' \sim t+\mu\left\{t\ln t-(d-1)\frac{(1-t)[1+(q-1)t]}{q}\ln\left(\frac{1-t}{1+(q-1)t}\right)\right\}. \tag{8}$$

The associated critical fixed point t^* satisfies

$$t^*\ln t^* = (d-1)\frac{(1-t^*)[1+(q-1)t^*]}{q}\ln\left(\frac{1-t^*}{1+(q-1)t^*}\right). \tag{9}$$

This equation yields the results presented in figure 6 as well as the following three results:

$$t^* \sim 1-q^{-1/(d-1)} \qquad d\to 1+0 \tag{10}$$

which coincides with the asymptotically exact result for the d-dimensional hypercubic lattice [8],

$$t^* = 1/(\sqrt{q}+1) \qquad d=2 \tag{11}$$

which coincides with the exact result for the square lattice, and

$$t^* \sim \exp[-(d-1)] \qquad d\to\infty \tag{12}$$

which differs from the exact result for the d-dimensional hypercubic lattice.

The fact that the $d\to 1$ result is asymptotically coincident with that of the d-dimensional hypercubic lattice arises from the fact that the linear chain has a special geometrical property, namely that it is simultaneously scale invariant (hierarchical lattice) and translationally invariant (Bravais lattice). The fact that the $d=2$ result exactly recovers that of the square lattice arises from the confluence of the diamond and trees transmissivities of the self-dual Wheatstone-bridge transmissivity. This is a

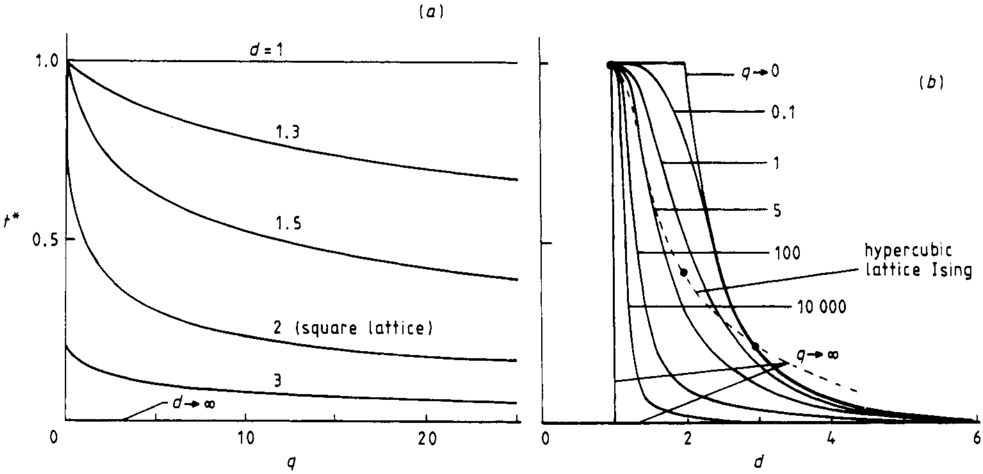

Figure 6. Dependence of the critical point t^* on q and d within the differential RG approach ($b'=1$ and $b\to 1$). (*a*) t^* against q for typical values of d (the $d=2$ curve coincides with the exact one for square lattice). (*b*) t^* against d for typical values of q (the results corresponding to the Ising ferromagnet in the hypercubic lattice have been included for comparison, the broken line being a guide to the eye).

manner for understanding why the differential Migdal-Kadanoff approach preserves self-duality.

From equation (8) we also obtain the thermal critical exponent

$$\nu^{-1}=1+\ln t^*-\frac{d-1}{q}\left[[q-2-2(q-1)t^*]\ln\left(\frac{1-t^*}{1+(q-1)t^*}\right)-q\right]. \quad (13)$$

This equation yields the results presented in figure 7 as well as the result

$$\nu\sim 1/(d-1) \qquad d\to 1+0 \quad (14)$$

which recovers the exact result for the d-dimensional hypercubic lattice [8] and the two results

$$\nu^{-1}=2[1-(1/\sqrt{q})\ln(\sqrt{q}+1)] \qquad d=2 \quad (15)$$

and

$$\nu\to 1 \qquad d\to\infty \quad (16)$$

which do not recover the exact results for the hypercubic lattice.

4. Conclusion

Let us summarise the main features of the present RG approach of the q-state Potts ferromagnet in hierarchical lattices. This approach is based on the renormalisation of b-sized two-rooted d-dimensional Migdal-Kadanoff-like graphs into b'-sized ones ($b'<b$). The results associated with $b'=1$ are, as usual, exact for the corresponding hierarchical lattices.

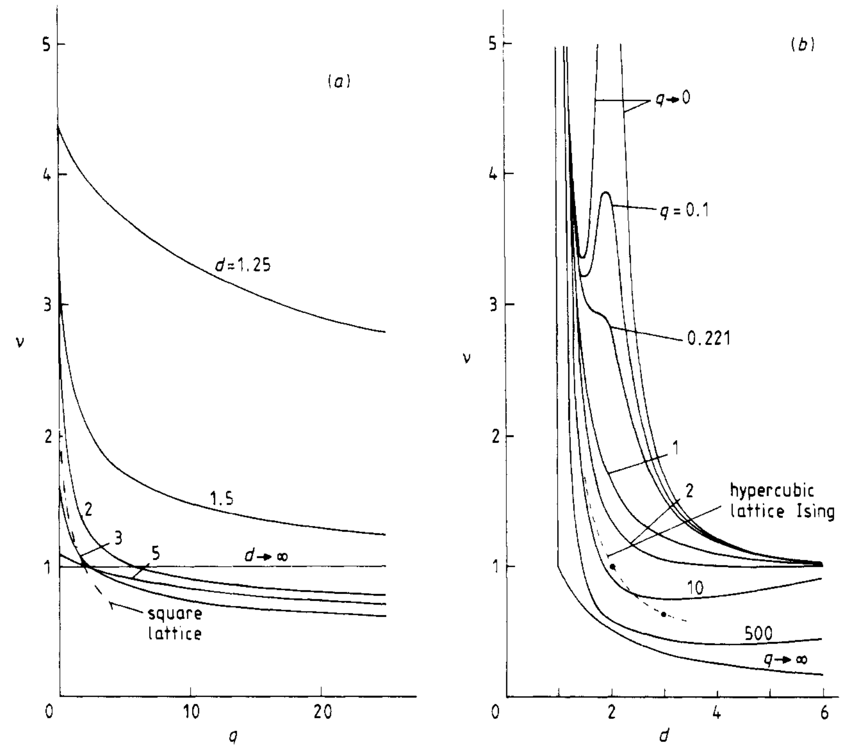

Figure 7. Dependence of the critical exponent ν on q and d within the differential RG approach ($b'=1$ and $b\to 1$). (a) ν against q for typical values of d (the exact result for square lattice has been included for comparison). (b) ν against d for typical values of q (the results corresponding to the Ising ferromagnet in hypercubic lattice have been included for comparison; the broken line being a guide to the eye).

Let us first stress an important point: transitions are, for all $d \geqslant 1$ and all $q \geqslant 0$, of the continuous type. This fact presents a remarkable discrepancy with Bravais lattices, which are known to yield first-order phase transitions for all $d > 1$ if q is high enough. In other words, the loss of the translational invariance of the system makes discontinuous phase transitions disappear.

Another interesting point is that, for fixed (b, b', d, q), the diamond and tress types present a different critical point but share one and the same value of ν. The d dependence of ν, at a fixed value of q, presents three different shapes according to whether q is in the interval $[0, q_{\min})$, $[q_{\min}, q_{\max}]$ or $(q_{\max}, \infty)$. In the first case ν presents a local minimum and a local maximum in the interval $1 < d \leqslant 2$, and monotonically decreases down to one for d increasing above 2. In the second case, ν monotonically decreases down to one for d increasing above one. In the third case, ν presents a minimum at a value of d which increases when q increases; also $\lim_{q\to\infty} \nu = 1/d$.

The $b\to\infty$ behaviours for t^* and ν are somewhat different from what is normally found for Bravais lattices. However, the reason for that might be not the loss of translational invariance but rather the fact that the critical temperature for the present cases is, in the $b\to\infty$ limit, not finite ($T_c = 0$ for diamonds, and $T_c \to \infty$ for tresses).

Finally, let us note that, in both $b' = b-1$ with $b\to\infty$ and $b'=1$ with $b\to 1$ cases, the linear expansion factor b/b' tends to unity. However, important differences are found for these two situations. For instance, in the former $t^* \to 0$ or 1 while in the latter t^* becomes a finite value between 0 and 1. In some sense, this type of discrepancy reinforces the well known fact that the knowledge of the intrinsic fractal dimensionality of an hierarchical lattice is nothing but one (though an important one) of the many ingredients which determine their criticality.

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